

# Random matrix model for antiferromagnetism and superconductivity on a two-dimensional lattice

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We suggest a new mean-field method for studying the thermodynamic competition between magnetic and superconducting phases in a two-dimensional square lattice. A partition function is constructed by writing microscopic interactions that describe the exchange of density and spin fluctuations. A block structure dictated by spin, time-reversal, and bipartite symmetries is imposed on the single-particle Hamiltonian. The detailed dynamics of the interactions are neglected and replaced by a normal distribution of random matrix elements. The resulting partition function can be calculated exactly. The thermodynamic potential has a structure which depends only on the spectrum of quasiparticles propagating in fixed condensation fields, with coupling constants that can be related directly to the variances of the microscopic processes. The resulting phase diagram reveals a fixed number of phase topologies whose realizations depend on a single coupling parameter ratio,  $\alpha$ . Most phase topologies are realized for a broad range of values of  $\alpha$  and can thus be considered robust with respect to moderate variations in the detailed description of the underlying interactions.

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## I. INTRODUCTION

Studies of high-temperature superconductors have revealed a rich phase diagram, with coexisting magnetic and superconducting correlations. These phase structures can be complex and include  $d$ -wave pairing, stripes, or the pseudogap phenomenon.<sup>1-4</sup> Theoretical models and numerical studies on a lattice indicate that the richness of the phase structure results from a delicate energy balance between competing states.<sup>2,5</sup> It follows that model predictions can be sensitive to small changes in the parameters of the theory or to details of the numerical approach. The question then arises of which properties of the phase diagram are constrained by the basic underlying symmetries and which are sensitive to the detailed dynamics of the interactions and to numerical approximations.

The purpose of this paper is to address this question with a new mean-field approach. The method is based on random matrix theory and consists in constructing a Hamiltonian that retains the basic spin, time-reversal, and bipartite symmetries of the problem but simplifies the dynamics of the interactions considerably. The theory is radically different from the familiar Hubbard or  $t$ - $J$  model. Here, we construct the model at a deeper microscopic level and describe interactions that are mediated by the exchange of density and spin fluctuations. This construction is inspired by random matrix models of the strong interaction, for which the QCD interactions are mediated by single-gluon exchange. Although natural in QCD, a microscopic description involving bosonic fields may be more controversial in the context of high- $T_c$  superconductors. Such an approach is similar to low-energy effective theories of antiferromagnets and superconductors<sup>6-10</sup> or to the antiferromagnetic spin-fluctuation exchange theory.<sup>11-14</sup> Here, in contrast to these models, no particular assumption is

made regarding the detailed dynamics of the exchange fields. Instead, we adopt a coarse description in which the block structure of the interaction matrix is dictated by the underlying symmetries of the Hamiltonian while individual matrix elements are drawn at random.

A random matrix approach offers three advantages. First, since the theory is constructed at a more microscopic level, it allows us to relate global properties of the phase diagram to specific microscopic mechanisms. Second, the simplified dynamics produces a mean-field model that can be solved exactly: the gap equations are polynomial. Their roots can therefore be studied as a function of the coupling parameters of the theory. Third, in the vicinity of critical points, the thermodynamic potential has a Landau-Ginzburg form in which the expansion coefficients satisfy specific symmetry constraints inherited from the deeper microscopic level. These constraints help us to identify those topologies that can be realized in the system and rule out those that violate the constraints.

The motivation for applying methods used in QCD to the high- $T_c$  problem follows from the strong analogies existing between these systems. First, the restoration of chiral symmetry with increasing quark density can be understood from the analogous behaviors of QCD and metamagnets. The chiral condensate plays the role of a staggered magnetization which vanishes abruptly as an external magnetic field is increased, driving the system through a first-order phase transition.<sup>15</sup> Second, single-gluon exchange is attractive in the antitriplet channel and leads to the Cooper pairing of quarks. This form of pairing can lead to a long-range order called *color superconductivity*.<sup>16-18</sup> The degrees of freedom that are involved in pairing are different from those involved in the chiral broken phase of QCD, so that color superconductivity competes directly with the breaking of chiral symmetry.

Random matrix methods have been extensively applied for studying the phase diagram of QCD.<sup>15,19–22</sup> In recent works,<sup>23–26</sup> we studied the phase diagram of QCD with three colors and two flavors as a function of temperature and quark chemical potential. The partition function was constructed as an integral over random matrices that mimicked the basic structure of quark-quark interactions but neglected their detailed dynamics. These matrices were given a block structure that reflected the spin, color, and flavor symmetries of one-gluon exchange. Inside a given block matrix, no further correlations were assumed among the matrix elements, which were then drawn at random on a normal distribution. This approach produced a mean-field model that could be solved exactly. The resulting effective potential gave polynomial gap equations whose roots could be determined analytically or numerically. The effective potential contained a single free parameter, defined as a coupling-constant ratio that measured the relative strength of the interactions in the chiral and diquark channels. As this coupling ratio was varied, the phase structure passed through a restricted number of distinct topologies. Moreover, starting with coupling constants with the values appropriate for single-gluon exchange, large variations were required to alter the topology of the phase diagram. We thus concluded that the QCD phase diagram was robust against moderate variations in the detailed dynamics of the interactions.

In general, random matrix models are useful in providing a global picture of the phase diagram. Being mean field in nature, such a picture is only a starting point that requires the proper inclusion of thermal, quantum, and spatial fluctuations (which play an important role in the high- $T_c$  problem as a consequence of the Mermin-Wagner theorem<sup>27</sup>) if it is to be quantitatively reliable. Nevertheless, the random matrix approach can be useful in providing an overview of the strength of the order parameters and their sensitivity to coupling parameters and can thus serve as a means for identifying those characteristics of the phase diagram that are protected by symmetry.

It is worth noting that the “random” part of the theory has nothing to do with disorder. Instead, the philosophy here consists in constructing a Hamiltonian with a block structure dictated by the underlying symmetries of the problem and replacing individual matrix elements by random variables, conventionally drawn on a normal distribution. This construction can be regarded as equivalent to integrating over many Hamiltonians that meet fundamental symmetry requirements but which differ from one another in the detailed implementation of the dynamics of the interaction. The theory can thus appear elaborate at first sight since it involves a large number of statistical variances. However, the number of free parameters decreases at each step of the calculations so that the theory becomes simpler as one proceeds toward the solution of the problem. In fact, the final form of the thermodynamic potential depends only on a single parameter ratio, and its functional form has a well-defined structure that can be interpreted in terms of quasiparticle energies. Hence, most of the effort involved in constructing the model is “upfront” but worthwhile since it provides relationships between the global phase diagram and the microscopic parameters of the theory. Overall, the procedure is relatively

simple and could be implemented in many other problems.

We will consider a fermion system on a two-dimensional square lattice and construct its partition function at finite temperature and finite chemical potential. In Sec. II, we show that extension of the methods used in QCD poses some challenges. First, while the basic interactions of QCD are naturally formulated in terms of quarks exchanging gluons, no such natural description is available at an elementary level for the cuprates. We will thus explicitly assume that the interactions can be described as the exchange of density and spin fluctuations. The structure of the interaction between the fermions and the fluctuation fields is dictated by SU(2)-spin, time-reversal, and bipartite symmetries. We will show that this formulation leads to a four-fermion effective potential whose terms can be compared to those of the Hubbard model. A second challenge is the need to account for the  $d$ -wave character of the superconducting order parameter, which forces us to introduce an explicit momentum dependence of the fermion states. As we are seeking to construct a model that suppresses as many details of the interaction as possible, we limit ourselves to a “coarse-grained” momentum description in which the first Brillouin zone is divided into four sectors in order to mimic the symmetry patterns of the antiferromagnetic and superconducting order parameters.

Section III is devoted to the derivation of the effective potential. We find that it is possible to construct a theory in which antiferromagnetism and  $d$ -wave superconductivity are favored and compete, while the  $s$ -wave channel is repulsive. Such a theory gives greater statistical weight to spin-fluctuation fields with a large momentum exchange. When deriving the effective potential, we will see that the theory simplifies considerably as one proceeds through the calculations. The initial model of Sec. II involves as many as eight different variance parameters; the final thermodynamic potential depends on a single parameter ratio. Its interpretation in terms of quasiparticle energies is also given in Sec. III.

We study the resulting phase diagrams in Sec. IV, where we show that, as was the case for QCD, there are only a finite number of possible topologies. As in the QCD problem, the topology of the phase structure changes gradually as the ratio of coupling constants is varied. Section V contains a summary of our main findings and our conclusions.

## II. CONSTRUCTION OF THE RANDOM MATRIX MODEL

We consider a system of electrons on a two-dimensional square lattice and model the competition between magnetic and superconducting orders as a function of the chemical potential,  $\mu$ , and the temperature,  $T$ . We construct interactions that satisfy spin, time-reversal, and bipartite symmetries with simplified but integrable dynamics. This approach ensures that the properties of the model arise solely as a consequence of the symmetries.

### A. Order parameters and the parametrization of momenta and frequencies

We wish to define random matrix correlators that mimic the basic structure of antiferromagnetism and superconduct-

ing order parameters. Working at finite temperature  $T$  in an imaginary time formalism, the antiferromagnetic order parameter assumes the form

$$\mathbf{m}_{\text{AF}} = \left\langle \sum_{\mathbf{p}, \omega_n, \alpha\beta} \psi_{\alpha}^{\dagger}(\mathbf{p} + \mathbf{Q}, \omega_n) \boldsymbol{\sigma}_{\alpha\beta} \psi_{\beta}(\mathbf{p}, \omega_n) \right\rangle, \quad (1)$$

where  $\mathbf{p}$  are momenta in the first Brillouin zone,  $\mathbf{Q} = (\pi\hbar/a, \pi\hbar/a)$  is the AF ordering vector ( $a$  is the lattice spacing),  $\omega_n = (2n+1)\pi T$  are fermion Matsubara frequencies,  $\alpha$  and  $\beta$  are spin indices,  $\boldsymbol{\sigma}$  are the spin Pauli matrices, and  $\langle \dots \rangle = \text{Tr}(\dots e^{-\beta H})$  denotes a thermal average. Similarly, the  $d$ -wave order parameter is given as

$$m_{\text{SC-}d} = \left\langle \sum_{\mathbf{p}, \omega_n} g(\mathbf{p}) \psi_{\uparrow}(\mathbf{p}, \omega_n) \psi_{\downarrow}(-\mathbf{p}, -\omega_n) \right\rangle, \quad (2)$$

where

$$g(\mathbf{p}) = \cos\left(\frac{p_x a}{\hbar}\right) - \cos\left(\frac{p_y a}{\hbar}\right) \quad (3)$$

is the  $d$ -wave form factor.

Our aim is to construct correlators that mimic the momentum couplings in Eqs. (1) and (2) on a coarse level. First, we divide the Brillouin zone into four regions related to one another by either a momentum shift in  $\mathbf{Q}$ ,

$$\{\mathbf{p}\} \mapsto Q\{\mathbf{p}\} = \{\mathbf{p} + \mathbf{Q}\}, \quad (4)$$

or by momentum reversal,

$$\{\mathbf{p}\} \mapsto P\{\mathbf{p}\} = -\{\mathbf{p}\}. \quad (5)$$

Next, we replace the exact form factor  $g$  in Eq. (2) by the simplified form factor

$$\phi_d(\mathbf{p}) = \text{sgn}[g(\mathbf{p})], \quad (6)$$

where  $\phi_d$  is a crude approximation of  $g$ . It neglects its variation with momentum, which is related to the detailed shape of the wave function, but exhibits the same  $d$ -wave symmetry as  $g$ , changing sign for every  $90^\circ$  rotation in the Brillouin zone. We believe that such an approximation captures the essential symmetry of the problem and is sufficient to describe  $d$ -wave pairing.

The particular form for  $\phi_d$  guides our parametrization of momentum states. Four momentum regions are chosen as the sectors in which  $\phi_d$  has a given sign. One possible division of the Brillouin zone is shown in Fig. 1. The approximate form factor  $\phi_d$  is  $-1$  in regions 1 and 3 and  $+1$  in regions 2 and 4. States in regions 1 and 3 are related to those in regions 2 and 4 by a shift of  $\pm\mathbf{Q}$ . Regions 1 and 2 are related to regions 3 and 4 by momentum reversal. Inside each region, states are labeled by an index  $i=1, \dots, M$ , where  $M$  scales with the total number of lattice sites and  $M \rightarrow \infty$  in the thermodynamical limit. These states also count the different Matsubara frequencies. From one region to another, the states are parametrized as follows. If a given index  $i$  refers to a state  $(\mathbf{p}, \omega_n)$  in region 1, then the states labeled with the index  $i$  in 2, 3, and 4, respectively, correspond to  $(\mathbf{p} + \mathbf{Q}, \omega_n)$ ,  $(-\mathbf{p}, -\omega_n)$ , and  $(-\mathbf{p} - \mathbf{Q}, -\omega_n)$ . (Note the change of sign in the frequencies of the last two terms.)

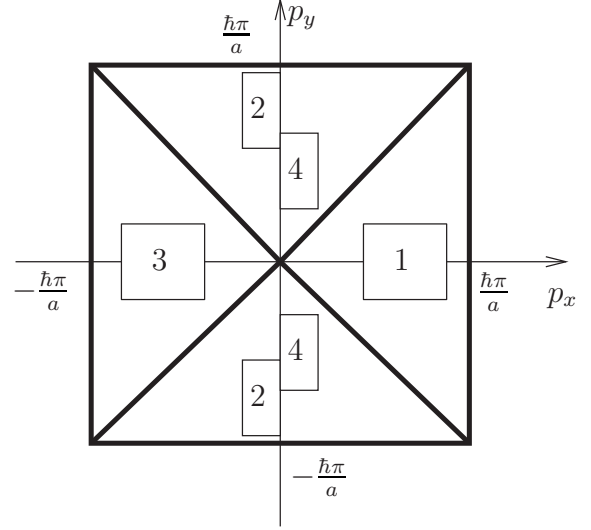


FIG. 1. Parametrization of the first Brillouin zone. Regions 1 and 2 are related by a momentum shift by  $\pm\mathbf{Q} = \pm(\pi\hbar/a, \pi\hbar/a)$ , as are regions 3 and 4. Regions 1 and 3 and regions 2 and 4 are related by momentum reversal.

With this parametrization, the AF order parameter is written as

$$\mathbf{m}_{\text{AF}} = \left\langle \sum_{r,s=1}^4 \sum_{i,j=1}^M \sum_{\alpha,\beta=\uparrow,\downarrow} \psi_{r,i,\alpha}^{\dagger} \boldsymbol{\sigma}_{\alpha\beta} (\Gamma_{\text{AF}})_{r,s} \delta_{ij} \psi_{s,j,\beta} \right\rangle, \quad (7)$$

where  $r$  and  $s$  are region indices and the four-by-four matrix  $\Gamma_{\text{AF}}$  couples momenta separated by  $\mathbf{Q}$ ,

$$\Gamma_{\text{AF}} = (\boldsymbol{\sigma}_1)_Q \otimes (\mathbf{1})_P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (8)$$

Similarly, the superconducting (SC) order parameter has the form

$$m_{\text{SC-}d} = \left\langle \sum_{r,s=1}^4 \sum_{i,j=1}^M \psi_{r,i,\uparrow} (\Gamma_{\text{SC-}d})_{r,s} \delta_{ij} \psi_{s,j,\downarrow} \right\rangle, \quad (9)$$

where  $\Gamma_{\text{SC-}d}$  is now blind to shifts by  $\mathbf{Q}$  and couples states with opposite momenta with a sign dictated by the  $d$ -wave form factor of Eq. (6)

$$\Gamma_{\text{SC-}d} = [\phi_d(\mathbf{p})]_Q \otimes (\boldsymbol{\sigma}_1)_P = (-\boldsymbol{\sigma}_3)_Q \otimes (\boldsymbol{\sigma}_1)_P = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (10)$$

Note that with our parametrization, the Kronecker symbol  $\delta_{ij}$  in Eqs. (7) and (9) automatically selects equal Matsubara frequencies for the AF order parameter and opposite frequencies for the SC order parameter.

Similar arguments can be followed for an  $s$ -wave order parameter. With an isotropic form factor  $\phi_s(\mathbf{p})=1$ , the corresponding momentum projector is given as

$$\Gamma_{\text{SC-}s} = (\mathbf{1})_Q \otimes (\sigma_1)_P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (11)$$

and the  $s$ -wave order parameter has the form given by Eq. (9) with  $\Gamma_{\text{SC-}d}$  replaced by  $\Gamma_{\text{SC-}s}$ .

### B. Constraints imposed by symmetries

We now turn to the construction of the random matrix interactions. Inspired by random matrix models for QCD,<sup>23,24</sup> we write the partition function of the system as a path integral

$$Z(\mu, T) = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi dH_{\text{int}} P(H_{\text{int}}) e^{-\psi^\dagger (H_0 + H_{\text{int}}) \psi}, \quad (12)$$

where  $\psi^\dagger$  and  $\psi$  are independent fermion fields,  $H_{\text{int}}$  is a matrix describing the random interaction with elements distributed according to the distribution  $P(H_{\text{int}})$  to be defined below, and  $H_0$  is the noninteracting part of the single-particle Hamiltonian which contains temperature and chemical-potential terms.

This formulation is radically different from the more familiar Hubbard or  $t$ - $J$  models, where the interaction terms are described by effective four-fermion potentials. Here, instead, the interactions are described at a deeper microscopic level so that fermions interact with fluctuation fields via current terms,  $\psi^\dagger H_{\text{int}} \psi$ . This formulation is directly inspired by the Yang-Mills Lagrangian for the strong interaction mediated by gluon exchange. In the context of the high- $T_c$  problem, these fluctuation fields can describe either interactions carried by phonons, antiferromagnetic fluctuations, or more complex effective interactions. Here, we will not attempt to identify the nature of these fields nor to specify their dynamics. We will assume only that interactions can be described in terms of such fields and consider how their description is constrained by the symmetries of the system.

#### 1. Interaction terms

According to the parametrization introduced in Sec. II A, the fermion fields are described by spinors with eight components (four momentum regions and two spins). Similarly, the random matrices  $H_{\text{int}}$  are composed of  $8 \times 8$  block matrices of size  $M \times M$ . Due to the symmetries of the system, these blocks are not completely independent.

Consider first the constraints imposed by SU(2). For a fixed pair of momentum region indices, e.g.,  $(r, s)$ , we can write

$$\begin{pmatrix} H_{\text{int},r,s,\uparrow,\uparrow} & H_{\text{int},r,s,\uparrow,\downarrow} \\ H_{\text{int},r,s,\downarrow,\uparrow} & H_{\text{int},r,s,\downarrow,\downarrow} \end{pmatrix} = \sum_{\mu=0}^3 \sigma_\mu H_{\mu;r,s}, \quad (13)$$

where  $\sigma_\mu = (\mathbf{1}, \boldsymbol{\sigma})$ ,  $H_{\mu;r,s}$  with  $\mu=0$  represents density-fluctuation fields, and  $H_{\mu;r,s}$  with  $\mu=1, 2, 3$  describes spin-

fluctuation fields. The interaction Hamiltonian can be made invariant under a spin unitary transformation,  $U$ , by requiring that the vector  $(H_{1;r,s}, H_{2;r,s}, H_{3;r,s})$  simultaneously undergoes a space rotation  $R$ , with  $R$  satisfying  $U^\dagger \sigma_i U = R_{ij} \sigma_j$ . The partition function itself, Eq. (12), is then made invariant under SU(2) transformations by requiring that the probability distribution,  $P(H_{\text{int}})$ , is invariant under the corresponding spatial rotations  $R$ .

Consider next constraints related to time-reversal invariance. Following the work of Monthoux on spin-fluctuation exchanges,<sup>13</sup> the fields  $H_\mu$  are taken to be real in coordinate representation—they do not carry an electric charge. As will be seen below, the integration over these fields produces a four-fermion interaction that contains squares of density terms,  $\sim (\psi^\dagger \psi)^2$ , and spin currents,  $\sim (\psi^\dagger \boldsymbol{\sigma} \psi)^2$ , thus leading to a two-body potential with a time-reversal symmetry. Now, because the fluctuation fields  $H_\mu$  are real, the Fourier components of their matrix elements must satisfy the constraint

$$(\mathbf{p}, \omega_n | H_\mu | \mathbf{q}, \omega_m) = (-\mathbf{p}, -\omega_n | H_\mu | -\mathbf{q}, -\omega_m)^*. \quad (14)$$

If one divides the Brillouin zone into two subspaces of states  $\{\mathbf{p}, \omega_n\}$  (regions 1 and 2) and  $\{-\mathbf{p}, -\omega_n\}$  (regions 3 and 4) and adopt the parametrization introduced in Sec. II A, this condition can be cast in the form

$$H_\mu = P H_\mu^* P, \quad (15)$$

where

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (16)$$

reverses both momentum and frequency. Hence, the matrices  $H_\mu$  must have the block structure

$$H_\mu = \begin{pmatrix} B_\mu & C_\mu \\ C_\mu^* & B_\mu^* \end{pmatrix}, \quad (17)$$

where  $B_\mu$  are Hermitian and  $C_\mu$  are complex symmetric.<sup>40</sup>

Finally, we turn to the bipartite symmetry, appropriate for a square lattice, composed of two interpenetrating sublattices  $A$  and  $B$ . Consider the transformation

$$\psi(\mathbf{r}) \mapsto \begin{cases} +\psi(\mathbf{r}) & \text{if } \mathbf{r} \in A, \\ -\psi(\mathbf{r}) & \text{if } \mathbf{r} \in B. \end{cases} \quad (18)$$

The full Hamiltonian  $H$  is not expected to be invariant under such a transformation because the kinetic terms couple fields defined on neighboring sites. However, we will assume that the *interaction* part of the Hamiltonian,  $H_{\text{int}}$ , is bipartite invariant (as is the case for the  $U$  term in the Hubbard model). We first determine the momentum representation of Eq. (18). Using the definition of  $\mathbf{Q} = (\pi, \pi)\hbar/a$ , Eq. (18) can be rewritten as  $\psi(\mathbf{r}) \mapsto \exp(i\mathbf{Q} \cdot \mathbf{r}/\hbar) \psi(\mathbf{r})$ . Then, in the momentum representation, the bipartite transformation takes the form

$$\psi(\mathbf{p}) \mapsto Q \psi(\mathbf{p}) = \psi(\mathbf{p} + \mathbf{Q}). \quad (19)$$

Dividing the Brillouin zone in the two subspaces  $\{\mathbf{p}\}$  (regions 1 and 3) and  $\{\mathbf{p} + \mathbf{Q}\}$  (regions 2 and 4), the bipartite invariance of  $H_\mu$  is written as



$$H_\mu = QH_\mu Q, \quad (20)$$

with

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (21)$$

$H_\mu$  must then have the block structure

$$H_\mu = \begin{pmatrix} D_\mu & E_\mu \\ E_\mu & D_\mu \end{pmatrix}, \quad (22)$$

where  $D_\mu$  and  $E_\mu$  are Hermitian. An alternative, but weaker, requirement on  $H_\mu$  could be that matrix elements between states with momenta  $\mathbf{p}_1 + \mathbf{Q}$  and  $\mathbf{p}_2 + \mathbf{Q}$  are equal to those between  $\mathbf{p}_1$  and  $\mathbf{p}_2$ . This amounts to take equal diagonal blocks in the right side of Eq. (22), with no constraint on the off-diagonal blocks. This second choice results in a free energy with a slightly different form but leads to the same main results as the choice of Eq. (22). This alternative form is discussed further in the Appendix.

Combining the requirements imposed by the three symmetries, we arrive at an interaction matrix of the form

$$H_{\text{int}} = \sum_{\mu=0}^3 \sigma_\mu \begin{pmatrix} B_{\mu d} & B_{\mu o} & C_{\mu d} & C_{\mu o} \\ B_{\mu o} & B_{\mu d} & C_{\mu o} & C_{\mu d} \\ C_{\mu d}^* & C_{\mu o}^* & B_{\mu d}^* & B_{\mu o}^* \\ C_{\mu o}^* & C_{\mu d}^* & B_{\mu o}^* & B_{\mu d}^* \end{pmatrix}, \quad (23)$$

where the  $4 \times 4$  block structure of  $H_{\text{int}}$  refers to the regions of the Brillouin zone. Each block is described by an  $M \times M$  matrix; blocks  $B_{\mu d}$  and  $B_{\mu o}$  are Hermitian, while  $C_{\mu d}$  and  $C_{\mu o}$  are complex symmetric. Overall,  $H_{\text{int}}$  contains 16 independent blocks.

## 2. Probability distribution

We represent the 16 blocks by  $A_b$  with  $b=1, \dots, 16$ . Their matrix elements are drawn on a normal distribution

$$P(H_{\text{int}}) = \exp\left(-8M \sum_{b=1}^{16} \sum_b^2 \text{Tr}(A_b A_b^\dagger)\right), \quad (24)$$

where  $\sum_b^2$  represent inverse variances. This form allows us to perform the integration over  $H_{\text{int}}$  analytically and thus to determine the partition function exactly.<sup>41</sup> In order to make the partition function invariant under  $SU(2)$  rotations, the inverse variances associated with each of the three blocks that describe a spin-fluctuation exchange are chosen equal. Since 12 of the 16 independent blocks describe spin fluctuations and four describe density fluctuations, we arrive at a total of  $4+4=8$  independent variances. The resulting distribution function is then given as

$$\begin{aligned} P(H_{\text{int}}) = \exp\{ & -8M[\sum_{B_{0d}}^2 \text{Tr}(B_{0d} B_{0d}^\dagger) + \sum_{\mathbf{B}_d}^2 \text{Tr}(\mathbf{B}_d \cdot \mathbf{B}_d^\dagger) \\ & + \sum_{B_{0o}}^2 \text{Tr}(B_{0o} B_{0o}^\dagger) + \sum_{\mathbf{B}_o}^2 \text{Tr}(\mathbf{B}_o \cdot \mathbf{B}_o^\dagger) \\ & + \sum_{C_{0d}}^2 \text{Tr}(C_{0d} C_{0d}^\dagger) + \sum_{\mathbf{C}_d}^2 \text{Tr}(\mathbf{C}_d \cdot \mathbf{C}_d^\dagger) \\ & + \sum_{C_{0o}}^2 \text{Tr}(C_{0o} C_{0o}^\dagger) + \sum_{\mathbf{C}_o}^2 \text{Tr}(\mathbf{C}_o \cdot \mathbf{C}_o^\dagger)]\}, \quad (25) \end{aligned}$$

where the inverse variances can be tuned individually at will

in order to favor various scattering mechanisms.

## 3. Noninteracting terms

The noninteracting part of the single-particle Hamiltonian is written as

$$\psi^\dagger H_0 \psi = \psi^\dagger (-\mu + \Omega_T + \Gamma_t) \psi. \quad (26)$$

Here, the chemical-potential term,  $\mu$ , is a scalar while  $\Omega_T$  and  $\Gamma_t$  are matrices which describe temperature dependence and hopping terms, respectively.

Temperature is introduced via Matsubara frequencies. Following previous work in QCD,<sup>23,24</sup> we include only the two lowest frequencies,  $\pm i\pi T$ . With this approximation,  $\Omega_T$  takes the form

$$\Omega_T = \text{diag}(i\pi T, -i\pi T) \otimes (\mathbf{1})_{\text{spin}} \otimes (\mathbf{1})_Q \otimes (\sigma_3)_P, \quad (27)$$

where the first term on the right side is an  $M \times M$  diagonal matrix. Hence, in each momentum region, half the states have a positive frequency while the other half have a negative frequency. The final term on the right side of Eq. (27) serves to implement the parametrization introduced above: states with a fixed label  $i$  in regions 1 and 2 have frequencies opposite to the corresponding states in regions 3 and 4.

Although limiting the sum over Matsubara frequencies leads to an oversimplified description of temperature dependence, we believe it to be sufficient to determine the general characteristics of the phase transition. In fact, we will see below that the parameter  $T$  serves as an energy scale which influences the energy balance between the various order parameters, much as the average thermal energy does when all frequencies are taken into account. The inclusion of all frequencies would certainly modify the resulting phase diagram but only in the trivial sense that every temperature  $T$  is mapped monotonically to a new value.<sup>28</sup> We will not seek to refine the  $T$  dependence here since such mapping does not alter the phase topology, i.e., the occurrence and the order of transition lines. Since  $T$  as introduced here is an arbitrary temperature scale, we will also drop the factor  $\pi$  to simplify notation.

The hopping term  $\Gamma_t$  in Eq. (26) is written as

$$\Gamma_t = \text{diag}(t, -t) \otimes (\mathbf{1})_{\text{spin}} \otimes (\sigma_3)_Q \otimes (\mathbf{1})_P, \quad (28)$$

where the first term on the right side is a diagonal  $M \times M$  matrix containing  $M/2$  elements of  $+t$  and  $M/2$  elements of  $-t$ . The matrix  $\Gamma_t$  mimics the nearest-neighbor hopping energy

$$\xi_{\mathbf{p}} = -2t_0[\cos(p_x a/\hbar) + \cos(p_y a/\hbar)], \quad (29)$$

whose band is symmetric around  $\xi_0=0$  and which satisfies  $\xi_{\mathbf{p}+\mathbf{Q}} = -\xi_{\mathbf{p}}$ . Here, again, we neglect the detailed momentum dependence of the hopping energy and retain only its symmetries around  $\xi_0=0$  and under a shift by  $\mathbf{Q}$ . In Eq. (28),  $t$  is a measure of the strength of the hopping term, which can be tuned against the variances and thus against the interaction strength.

## III. THERMODYNAMIC POTENTIAL

In this section we evaluate the thermodynamic potential corresponding to the partition function of Eq. (12) using

methods which are standard in random matrix theory.<sup>19,23</sup> It is worth emphasizing that the random matrix model of Eq. (12) is solvable, i.e., the saddle-point method to be discussed below becomes exact in the thermodynamic limit. We will concentrate on the main results and present more detailed calculations in the Appendix. These calculations are performed in three steps. The first step consists of integrating over the interaction matrix elements according to the distribution  $P(H_{\text{int}})$  of Eq. (25). This integration leads to a four-fermion interaction,  $Y$ , which receives contributions from the eight independent blocks of  $H_{\text{int}}$  in Eq. (23). In the second step, the fermion fields are rearranged and the terms of  $Y$  are put in two groups. The first group contains products of bilinears of the form  $\sim \psi_i^\dagger \psi_i$  (with an implicit sum over  $i$ ), which includes the antiferromagnetic order parameter. The second group contains products involving bilinears of the form  $\psi_i \psi_i$ , which are relevant for superconductivity. To keep track of momentum and spin indices while rearranging terms, we use the generalized Fierz identities of the Appendix. In the final step, quartic fermion terms are linearized by means of Hubbard-Stratonovitch transformations, which introduce an auxiliary real field  $\sigma$ , to be associated with an antiferromagnetic order parameter, and a complex field  $\Delta$ , to be associated with superconductivity. The resulting form of the partition function is given as

$$Z(\mu, T) = \int d\sigma d\Delta d\Delta^* e^{-8M\Omega(\sigma, \Delta)}, \quad (30)$$

where  $\Omega$  is the thermodynamic potential

$$\Omega(\sigma, \Delta) = A|\Delta|^2 + B\sigma^2 - \frac{1}{4} \log[(\sqrt{\sigma^2 + t^2} - \mu)^2 + |\Delta|^2 + T^2] - \frac{1}{4} \log[(\sqrt{\sigma^2 + t^2} + \mu)^2 + |\Delta|^2 + T^2], \quad (31)$$

with

$$A \equiv 8 \left( \frac{1}{\Sigma_{B0d}^2} - \frac{1}{\Sigma_{B0o}^2} + \frac{1}{\Sigma_{C0d}^2} - \frac{1}{\Sigma_{C0o}^2} - \frac{3}{\Sigma_{B_d}^2} + \frac{3}{\Sigma_{B_o}^2} - \frac{3}{\Sigma_{C_d}^2} + \frac{3}{\Sigma_{C_o}^2} \right)^{-1}, \quad (32)$$

$$B \equiv 8 \left( -\frac{1}{\Sigma_{B0d}^2} - \frac{1}{\Sigma_{B0o}^2} - \frac{1}{\Sigma_{C0d}^2} - \frac{1}{\Sigma_{C0o}^2} + \frac{1}{\Sigma_{B_d}^2} + \frac{1}{\Sigma_{B_o}^2} + \frac{1}{\Sigma_{C_d}^2} + \frac{1}{\Sigma_{C_o}^2} \right)^{-1}. \quad (33)$$

Equations (31)–(33) constitute the main result of the model. Note that the factor  $8M$  in the argument of the exponential in Eq. (30) plays the role of the volume of the system since  $M$  scales with the total number of lattice sites. The phase diagram can be established with the help of the saddle-point approximation to Eq. (30), which requires the simultaneous solution of the two gap equations  $\partial\Omega/\partial\sigma=0$  and  $\partial\Omega/\partial\Delta=0$ . Due to the logarithm term in the right side of Eq. (31), these equations are polynomial in the auxiliary fields and can be solved analytically. In the thermodynamic limit  $M \rightarrow \infty$ , the saddle-point approximation becomes exact and the solutions

of the gap equations that achieve the lowest value for  $\Omega$  describe the thermodynamic phases of the system.

As mentioned in Sec. I, the model becomes increasingly simple as one proceeds through the calculation. From the initially large number of parameters required to describe  $H_{\text{int}}$  in Eq. (23), there remain only a few constants in the final form of the thermodynamic potential [Eq. (31)]. This potential has a remarkably simple structure. Anticipating the results of Sec. IV, we note first that the topology of the phase diagram depends only on a single parameter ratio

$$\alpha = \frac{B}{A}, \quad (34)$$

whose strength characterizes the relative importance of superconductivity and antiferromagnetism. The terms  $A|\Delta|^2$  and  $B\sigma^2$  represent the energy cost of creating a constant field in the corresponding channel. As shown in the Appendix, the logarithmic term in  $\Omega$  corresponds to the determinant of the Hamiltonian for a single fermion in fixed constant external fields  $\sigma$  and  $\Delta$ . These terms have the generic form

$$\sim \sum_{\varepsilon_{\pm}} \log(iT - \varepsilon_{\pm})(-iT - \varepsilon_{\pm}), \quad (35)$$

where  $\varepsilon_{\pm}$  are the quasiparticle energies

$$\varepsilon_{\pm} = [(\sqrt{t^2 + \sigma^2} \pm \mu)^2 + |\Delta|^2]^{1/2}. \quad (36)$$

These expressions are strongly reminiscent of the quasiparticle energies of earlier mean-field models.<sup>29,30</sup> Neglecting the triplet order and approximating the square of the form factor as  $\phi_d^2 \approx 1$  in Eqs. (10) and (11) of Ref. 29, one finds quasiparticle energies of the form  $E_{\pm}(\mathbf{p}) = [(\sqrt{\xi_{\mathbf{p}}^2 + \sigma^2} \pm \mu)^2 + |\Delta|^2]^{1/2}$ , with  $\sigma = 2Jm$  and  $\Delta = Jd$ . Equation (36) shows a result of similar structure with, however, the simplification  $\xi_{\mathbf{p}} \sim t$  which is a consequence of our coarse description of momentum states which leads us to ignore the detailed momentum dependence of kinetic-energy terms.

Thus, the basic structure of the potential  $\Omega$  is simply related to the energies of the elementary excitations of the system for fixed constant external fields. This means that the thermodynamic potential could have been constructed immediately from the knowledge of the quasiparticle energies alone, without going through the steps described in Sec. II. Note, however, that the additional information which comes from constructing the interactions at the more microscopic level is useful. Through the dependence of  $A$  and  $B$  on the individual variances, it establishes connections between the microscopic mechanisms and the global properties of the system.

#### IV. PHASE DIAGRAM

We now consider the various topologies that can be realized in the phase diagram. Despite the simplicity of the thermodynamic potential, a full exploration of the parameter space is a considerable task. Thus, we concentrate on a restricted number of physically relevant cases.

First, we assume that the interactions are attractive in the antiferromagnetic channel. As shown in the Appendix, this

requires that the variances satisfy the inequality

$$B = 8 \left( -\frac{1}{\Sigma_{B_0d}^2} - \frac{1}{\Sigma_{B_0o}^2} - \frac{1}{\Sigma_{C_0d}^2} - \frac{1}{\Sigma_{C_0o}^2} + \frac{1}{\Sigma_{B_d}^2} + \frac{1}{\Sigma_{B_o}^2} + \frac{1}{\Sigma_{C_d}^2} + \frac{1}{\Sigma_{C_o}^2} \right)^{-1} > 0, \quad (37)$$

which implies that spin-fluctuation exchange is stronger than density-fluctuation exchange. In fact, this condition can be related to the requirement that the on-site potential in the Hubbard model is repulsive. In momentum representation, the Hubbard Hamiltonian is written as

$$H_H = \sum_{\alpha\mathbf{p}} \varepsilon_{\mathbf{p}} \psi_{\alpha\mathbf{p}}^\dagger \psi_{\alpha\mathbf{p}} + U \sum_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4} \delta_{\mathbf{p}_1+\mathbf{p}_3,\mathbf{p}_2+\mathbf{p}_4} \psi_{\uparrow\mathbf{p}_1}^\dagger \psi_{\uparrow\mathbf{p}_2} \psi_{\downarrow\mathbf{p}_3}^\dagger \psi_{\downarrow\mathbf{p}_4}, \quad (38)$$

with  $U$  positive. The corresponding partition function contains the thermal average  $\text{Tr}(e^{-H_H/T} \dots)$ , where the equivalent of the weighting factor  $e^{-H_H/T}$  in the random matrix model is the term  $e^Y$ . Hence, a *positive*  $U$  in the Hubbard model would correspond to a *negative* four-fermion term of the form  $\psi_{\uparrow i}^\dagger \psi_{\uparrow j} \psi_{\downarrow j}^\dagger \psi_{\downarrow i}$  in the random matrix model. Close inspection of the four-fermion potentials in Eqs. (A3)–(A10) reveals that they contain such terms and that they will be negative provided the inequality of Eq. (37) is satisfied.

As a second assumption, we consider interactions which are attractive in the superconducting  $d$ -wave channel but repulsive in the  $s$ -wave channel. This requires a particular choice of the inverse variances. In the Appendix, we show that such an interaction can be found among those that favor spin fluctuations by putting more statistical weight on the blocks  $\mathbf{B}_o$  and  $\mathbf{C}_o$  than on  $\mathbf{B}_d$  and  $\mathbf{C}_d$ . This choice leads to a positive constant  $A$  in Eq. (32). Such a choice of the variances is expected to result from interactions which favor large momentum transfer of order  $\sim \mathbf{Q}$ . This result can be related to the antiferromagnetic spin-fluctuation model of Ref. 14 in which the spin susceptibility is peaked at a momentum exchange  $\sim \mathbf{Q}$ . In the present approach, however, we do not attempt to describe the dynamics of spin exchange in detail but rather use the coarse device of the inverse variances to tune the relative strengths of the exchange mechanisms.

Given the restricted parameter space that results from these two assumptions, the system can develop different phase structures in the  $(\mu, T)$  plane as a function of the parameter ratio  $\alpha=B/A$ . The phase structures can be grouped according to their topology. We identify four distinct topologies which emerge as the parameter  $\alpha$  is gradually increased. The system switches from one topology to the next at specific values of  $\alpha$  that depend on the strength of the hopping term  $t$ . In general, systems with larger  $t$  develop a larger kinetic energy per charge carrier and are found to favor superconductivity over antiferromagnetism.

It is useful to note that because our model only includes nearest-neighbor hopping, it cannot distinguish between hole- and electron-doped systems. This symmetry can be seen in the potential of Eq. (31), where  $\Omega$  is an even function of  $\mu$ . Hence,  $\mu=0$  corresponds here to half filling. In prin-

ciple, next-nearest neighbor and higher-order hopping terms could be added to the model by using a hopping matrix,  $\Gamma$ , whose eigenvalues have a sign distribution that reproduces the symmetries of the corresponding kinetic energies in the various regions of the Brillouin zone. Although we have limited ourselves to nearest-neighbor terms here, we expect that a model with a more elaborate hopping matrix would distinguish between electron and hole dopings.

### A. Antiferromagnetism alone

For the smallest values of  $\alpha=B/A$ , superconductivity is too weak to compete with antiferromagnetism. Such a situation corresponds, for instance, to a large value for  $A$ , leading to a prohibitively large energy cost  $\sim A|\Delta|^2$  for creating a constant field  $\Delta$ .

Consider first the system at zero temperature and at half filling ( $\mu=0$ ). Setting  $\Delta=0$  in Eq. (31), we find the gap equation

$$\left. \frac{\partial \Omega}{\partial \sigma} \right|_{\Delta=0, \mu=0, T=0} = 2B\sigma - \frac{\sigma}{\sigma^2 + t^2} = 0, \quad (39)$$

which gives either  $\sigma=0$  [paramagnetic phase (PA)] or  $\sigma = \sqrt{1/(2B) - t^2}$  [antiferromagnetic phase (AF)]. The latter solution is real if the hopping term is not too strong,  $2Bt^2 < 1$ . Given that  $B$  scales as an inverse variance [see Eq. (33)], this condition is equivalent to the inequality  $t/t_{\text{TH}} < 1$ , where the threshold value  $t_{\text{TH}} = 1/\sqrt{2B}$  is a measure of the interaction strength. When this condition is fulfilled,  $\sigma = \sqrt{1/(2B) - t^2}$  is the absolute minimum of  $\Omega$  and the ground state is antiferromagnetic. In the rest of this work, we will explicitly assume that  $t < t_{\text{TH}}$ , so that the half-filled state is antiferromagnetic.

In contrast to our approach, mean-field studies of the Hubbard model find an antiferromagnetic ground state no matter how weak the interactions are or how large the hopping term is.<sup>31,32</sup> There, the absence of a threshold results from the logarithmic singularity that occurs in the density of states at the edge of the magnetic Brillouin zone. At half filling, the divergent density of states leads to a gap with an exponential dependence on  $t/U$ ,  $\sigma \sim t \exp(-2\pi\sqrt{t/U})$ , where  $U$  is the strength of the on-site repulsion. In the random matrix approach, the detailed band structure is ignored and, because of the coarse description of momentum states, no divergence appears in the density of states. As a result, the interaction must be sufficiently strong to produce an antiferromagnetic ground state. Given the level of approximations underlying our approach, this behavior is not unreasonable. Discrepancies with microscopic theories are to be expected in cases where the condensates are weak and thus sensitive to fluctuations. Understanding the fate of these condensates clearly requires more than mean-field approximation.

We now turn to the phase diagram in the  $(\mu, T)$  plane. Choosing  $t/t_{\text{TH}}=0.5$ , the superconducting phase does not develop so long as  $\alpha < \alpha_\Delta$  with  $\alpha_\Delta \approx 0.1$ . The corresponding phase structure is shown in Fig. 2 and resembles that of chiral symmetry breaking in QCD with two flavors and three colors, in the limit where a color-superconducting phase is ignored (see Ref. 23). The gap equation  $d\Omega/d\sigma=0|_{\Delta=0}$  has a

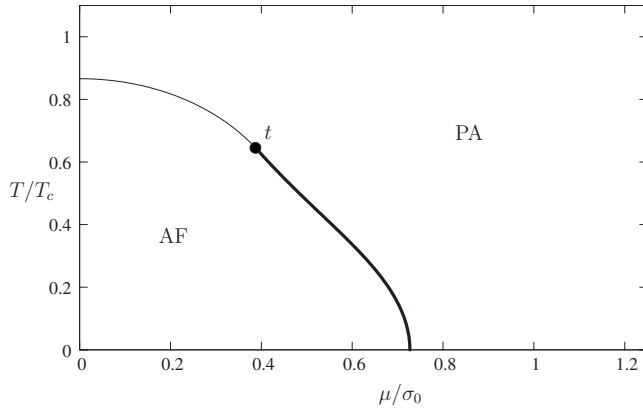


FIG. 2. Phase diagram for  $t/t_{\text{TH}}=0.5$  and  $\alpha<0.1$ . The transition from the AF to the PA is second order at half filling and first order at zero temperature (thick line). These two lines merge at a tricritical point,  $t$ . Here, temperature is plotted in units of  $T_c \equiv T_c(\mu=0, t=0)$ , which is the transition temperature at half filling in the limit  $t \rightarrow 0$ , while chemical potential is plotted in units of  $\sigma_0 \equiv 1/\sqrt{2}B$ , which represents the AF field at half filling, zero temperature, and for  $t \rightarrow 0$ .

form similar to the QCD problem. It has a solution with  $\sigma=0$ , describing a paramagnetic phase, and solutions which satisfy the quadratic equation

$$x^2 + 2x \left( -\mu^2 + T^2 - \frac{1}{4B} \right) + (\mu^2 + T^2)^2 + \frac{\mu^2 - T^2}{2B} = 0, \quad (40)$$

where  $x = \sigma^2 + t^2$ . For moderate  $\mu$  and high temperature, the only real solution is  $\sigma=0$  and the system is in a paramagnetic phase. Decreasing  $T$  at fixed  $\mu$ , two additional real solutions,

$$\sigma = \pm \sigma_{\text{AF}} = \pm \left( \mu^2 - t^2 - T^2 + \frac{1}{4B} + \frac{1}{4B} \sqrt{1 - 64B^2 \mu^2 T^2} \right)^{1/2}, \quad (41)$$

can be found below the critical temperature

$$T_c(\mu, t) = \left( \frac{1}{4B} - \mu^2 - t^2 + \frac{1}{4B} \sqrt{1 - 16B\mu^2 + 64B^2\mu^2 t^2} \right)^{1/2}. \quad (42)$$

Below  $T_c(\mu, t)$ , these solutions are local minima of the free energy  $\Omega(\sigma, \Delta=0)$  and  $\sigma=0$  becomes a maximum. The finite roots  $\pm \sigma_{\text{AF}}$  describe an antiferromagnetic phase. They vanish at  $T_c(\mu, t)$ , which thus characterizes a second-order transition from an antiferromagnetic to a paramagnetic phase. In the opposite regime of low temperatures and finite  $\mu$ , the transition is found to be discontinuous. It takes place along a first-order line [actually, a triple line (see Ref. 23)] which starts on the zero-temperature axis at

$$\mu_1 = \sqrt{\frac{0.14}{B} + t^2} \quad (43)$$

and extends with decreasing  $\mu$  toward the zero- $\mu$  axis. This line meets the second-order line  $T=T_c(\mu, t)$  at the tricritical point  $t \equiv (\mu_3, T_3)$ , given as

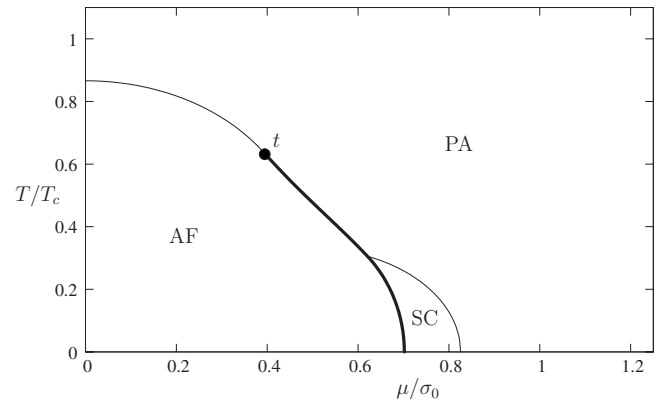


FIG. 3. Phase diagram for  $t/t_{\text{TH}}=0.5$  and  $\alpha=0.2$ . In addition to the phase structure of Fig. 2, there is a SC emerging out of the antiferromagnetic phase via a first-order transition. This phase undergoes a second-order transition to the paramagnetic phase at either higher  $\mu$  or higher  $T$ . Here,  $t$  is a tricritical point. The scales  $T_c$  and  $\sigma_0$  are those defined in the caption of Fig. 2.

$$\mu_3 = \left( -\frac{1 - 4Bt^2}{8B} + \frac{\sqrt{1 + (1 - 4Bt^2)^2}}{8B} \right)^{1/2}, \quad (44)$$

$$T_3 = \left( \frac{1 - 4Bt^2}{8B} + \frac{\sqrt{1 + (1 - 4Bt^2)^2}}{8B} \right)^{1/2}. \quad (45)$$

The first- and second-order lines meet with equal slopes  $dT/d\mu$ .

The thermodynamic potential  $\Omega(\sigma, \Delta=0)$  can be expanded as a series of powers in  $\sigma$  near the critical lines. The result resembles a Ginzburg-Landau expansion. For  $\mu < \mu_3$  and near  $T_c(\mu, t)$ , the free energy is found to scale as  $\Omega(\sigma, \Delta=0) \approx \Omega(\sigma=0, \Delta=0) + b_4(\mu, T)\sigma^4 + \mathcal{O}(\sigma^6)$ , so that the second-order phase transition has the critical exponents of a mean-field  $\phi^4$  theory. Near the tricritical point, one has  $\Omega(\sigma, \Delta=0) \approx \Omega(\sigma=0, \Delta=0) + b_6(\mu, T)\sigma^6 + \mathcal{O}(\sigma^8)$  and the critical exponents are now those of a mean-field  $\phi^6$  theory. Note, however, that the coefficients  $b_4$  and  $b_6$  here are *known* functions of  $\mu$  and  $T$ .

### B. Competition between antiferromagnetism and superconductivity

As the coupling ratio  $B/A$  increases, a superconducting phase can be favored over an antiferromagnetic one. As doping is increased, the new carriers disrupt the antiferromagnetic correlations. If the interaction is sufficiently strong in the pairing channel, this can lead to a transition to a superconducting phase with  $\Delta \neq 0$ . Such a transition can generally take place in one of two ways, either through the appearance of a “wedge” of mixed broken symmetry with both  $\sigma \neq 0$  and  $\Delta \neq 0$ , with continuous transitions toward the pure AF and SC phases, or via a discontinuous transition between the two pure phases. The first case was encountered in a random matrix model of Ref. 23 when the coupling ratio of QCD was altered in favor of color superconductivity. Here, however, the second case is found as shown in Fig. 3.

To understand the onset of superconductivity, consider a pure phase with  $\sigma=0$  and  $\Delta \neq 0$ . The gap equation,  $d\Omega(\sigma$



$=0, \Delta)/d\Delta=0$  always has a root  $\Delta=0$ , which is a local minimum at either large  $\mu$  or large  $T$ . For moderate  $\mu$  and for  $T$  less than

$$T_{c\Delta}(\mu, t) = \left( \frac{1}{4A} - \mu^2 - t^2 + \frac{1}{4A} \sqrt{1 + 64A^2 \mu^2 t^2} \right)^{1/2}, \quad (46)$$

the solution  $\Delta=0$  becomes a local maximum, while  $\Omega(0, \Delta)$  exhibits two local minima, given by the roots with

$$|\Delta| = \Delta_{SC} = \left( \frac{1}{4A} - \mu^2 - T^2 - t^2 + \frac{1}{4A} \sqrt{1 + 64A^2 \mu^2 t^2} \right)^{1/2}, \quad (47)$$

which describe the superconducting phase.  $\Delta_{SC}$  vanishes on the curve  $T=T_{c\Delta}(\mu, t)$ , which is thus a second-order transition line. In particular, the thermodynamic potential behaves like  $\Omega(\sigma=0, \Delta) \approx \Omega(0, 0) + a_4 |\Delta|^4 + \mathcal{O}(|\Delta|^6)$  in the vicinity of the phase boundary.

The curve  $T=T_{c\Delta}(\mu, t)$  meets the  $T=0$  axis at  $\mu_+ = [1/(4A) + t^2 + \sqrt{1 + 16At^2/(4A)}]^{1/2}$ . For  $t^2 \leq 1/(2A)$ , this line also meets the  $\mu=0$  axis, so that the region  $T \leq T_{c\Delta}(\mu, t)$  contains the half-filled state with  $\mu=T=0$ . For  $t^2 > 1/(2A)$ , on the other hand, the curve  $T=T_{c\Delta}(\mu, t)$  has the shape of a dome, which starts on the  $T=0$  axis at  $\mu_- = [t^2 + 1/(4A) - \sqrt{1 + 16At^2/(4A)}]^{1/2}$ , reaches a maximum  $T_{\max} = [1/(8At)] \sqrt{1 + 16At^2}$  at  $\mu_{\max} = \sqrt{64A^2 t^4 - 1}/(8At)$ , and decreases again to meet the  $T=0$  axis at  $\mu = \mu_+$ .

The condition for the superconducting phase to emerge out of the AF phase is now clear. The curve  $T=T_{c\Delta}$  must end at  $\mu_+ > \mu_1$ , where  $\mu_1$  is given by Eq. (43). In that case, there is an intermediate region where the superconducting state achieves a lower energy than both the paramagnetic and the antiferromagnetic states. For a fixed ratio  $t/t_{TH}$ , the threshold condition  $\mu_+ = \mu_1$  gives the critical value  $\alpha = \alpha_\Delta$  which marks the onset of superconductivity. The result is a decreasing function of  $t$ , with  $\alpha_\Delta \approx 0.28$  for  $t=0$ ,  $\alpha_\Delta \approx 0.1$  for  $t=0.5t_{TH}$ , and  $\alpha_\Delta \approx 0.03$  in the limit  $t \rightarrow t_{TH}$ . For values of  $\alpha > \alpha_\Delta$ , the superconducting phase develops in a wedge adjacent to the antiferromagnetic phase, with a first-order transition toward the antiferromagnetic state at the lower  $\mu$  and a second-order transition toward the paramagnetic phase at the higher  $\mu$ , as illustrated in Fig. 3.

Figure 4 shows the zero-temperature auxiliary fields  $\sigma_{AF}$  and  $\Delta_{SC}$  as a function of the chemical potential,  $\mu$ . Note that  $\Delta_{SC}$  vanishes at  $\mu = \mu_+$  with the mean-field exponent 1/2 so that  $\Delta_{SC} \sim (\mu_+ - \mu)^{1/2}$ . The variation of the antiferromagnetic field  $\sigma_{AF}$  with  $\mu$  should not be considered significant; it is a direct consequence of the approximate description of temperature dependence. In fact, taking the sum over all Matsubara frequencies produces a constant condensation field by a mechanism similar to that observed in the phase diagram of QCD with two colors and light masses (see Ref. 28).

### C. Bicritical point

As  $\alpha = B/A$  increases above  $\alpha_\Delta$ , the superconducting phase boundary slides up along the first-order transition line between the antiferromagnetic and the paramagnetic phases. At a new critical value,  $\alpha = \alpha_b$ , the superconducting phase

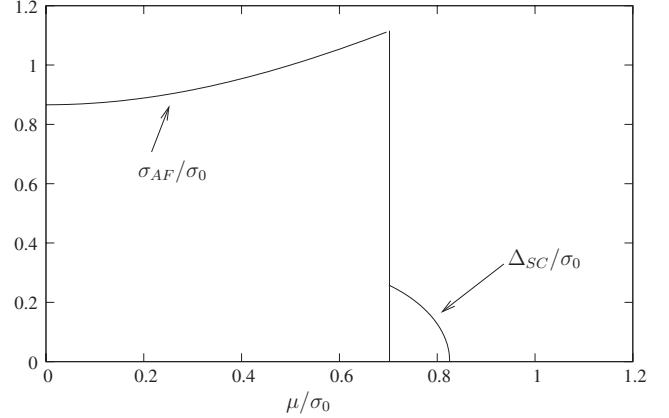


FIG. 4. Zero-temperature auxiliary fields as a function of  $\mu$  for the parameters corresponding to the phase diagram of Fig. 3. The scale  $\sigma_0$  is defined in the caption of Fig. 2.

boundary reaches the tricritical point. The value of  $\alpha_b$  is readily determined from the condition  $T_{c\Delta}(\mu_3, t) = T_3$ , where  $\mu_3$ ,  $T_3$ , and  $T_{c\Delta}$  are, respectively, given by Eqs. (44)–(46). For  $t \rightarrow 0$ , this gives  $\alpha_b = \sqrt{2}/2$ ;  $\alpha_b$  then decreases with  $t$ , is equal to  $\alpha_b \approx 0.62$  for  $t = 0.5t_{TH}$ , and reaches  $\alpha_b \approx 0.29$  in the limit  $t \rightarrow t_{TH}$ .

For  $\alpha > \alpha_b$ , the topology of the phase structure is changed. The two second-order lines  $T_c(\mu, t)$ , Eq. (42), and  $T_{c\Delta}(\mu, t)$ , Eq. (46), now intersect at a new critical point,  $b$ , with

$$\mu_b = \left( \frac{\alpha}{4B} - \frac{\alpha^2}{4B} + t^2(1 - \alpha^2) \right)^{1/2}, \quad (48)$$

$$T_b = \left( \frac{\alpha}{4B} + \frac{\alpha^2}{4B} - \alpha^2 t^2 \right)^{1/2}. \quad (49)$$

In the vicinity of  $b$ , the thermodynamic potential can be expanded as  $\Omega(\sigma, \Delta) \approx \Omega(0, 0) + a_4 |\Delta|^4 + b_4 \sigma^4 + c_4 \sigma^2 |\Delta|^2$ , where  $a_4$ ,  $b_4$ , and  $c_4$  are known coefficients satisfying the inequality  $4a_4 b_4 - c_4^2 < 0$ . As a result, the global minimum of  $\Omega$  can only be realized by a pure phase—either paramagnetic, antiferromagnetic, or superconducting. A mixed broken-symmetry state with both nonvanishing  $\sigma$  and  $\Delta$  cannot be a global minimum. Therefore,  $(\mu_b, T_b)$  is a bicritical point which ends a first-order line separating the antiferromagnetic and superconducting phases. The coupled gap equations actually have a root with a mixed broken symmetry. We have verified that, away from  $b$  and at lower temperatures, (i) this state is always metastable (i.e., it is a saddle point) whenever its fields are real and (ii) the transition line between the antiferromagnetic and the superconducting phases is first order all the way to the  $T=0$  axis. The resulting phase structure is shown in Fig. 5 for the case  $t=0.5t_{TH}$  and  $\alpha=0.8$ .

For even larger values of  $\alpha$ , the bicritical point migrates toward the  $\mu=0$  axis and reaches it when  $\alpha=1$ . In this case,  $T_b = T_c(\mu=0, t)$ . The antiferromagnetic phase thus exists only at half filling, and the superconducting phase develops for all finite  $\mu$  in the region  $T < T_{c\Delta}(\mu, t)$ . For  $\alpha > 1$ , the superconducting phase wins over the antiferromagnetic state, which disappears from the phase diagram.

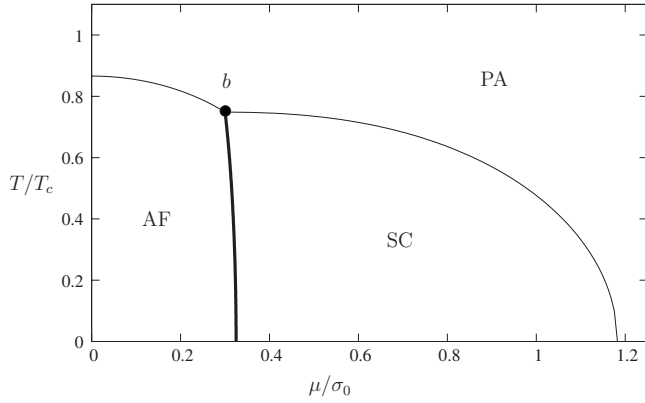


FIG. 5. Phase diagram for  $t/t_{\text{TH}}=0.5$  and  $\alpha=0.8$ . The superconducting phase has grown beyond the tricritical point of Fig. 3. The two second-order lines (thin lines) separating, respectively, the antiferromagnetic and superconducting phases from the paramagnetic phase now meet at a bicritical point,  $b$ , which is also the end point of a first-order line (thick line) between the AF and SC phases. The scales  $T_c$  and  $\sigma_0$  are those defined in the caption of Fig. 2.

## D. Discussion

### 1. Symmetries of $H_{\text{int}}$ and correlations

The interaction part of the Hamiltonian was constructed by imposing three different symmetries, which provide the correlations necessary for generating finite condensates. Spin rotational and time-reversal symmetries are intimately related to magnetism.<sup>39</sup> The spin symmetry is obviously necessary in order to express the order parameters. Time-reversal symmetry leads to a block structure which produces terms of the form  $\psi_i^\dagger \psi_j \psi_i^\dagger \psi_j$ , thereby yielding pairing condensates  $\sim \psi_j \psi_j$  and  $\psi_i^\dagger \psi_i^\dagger$ . The bipartite block structure induces correlations among states whose momenta are separated by  $\mathbf{Q}$ . The resulting four-fermion potential,  $\sim \psi_i^\dagger \psi_j \psi_j^\dagger \psi_i$ , thus yields condensates of the form  $\sim \psi_i^\dagger \psi_i$  where the coupled momenta are separated by  $\mathbf{Q}$ .

### 2. Effect of the hopping term

Since the form of the thermodynamic potential is relatively simple, it is easy to vary the hopping parameter in

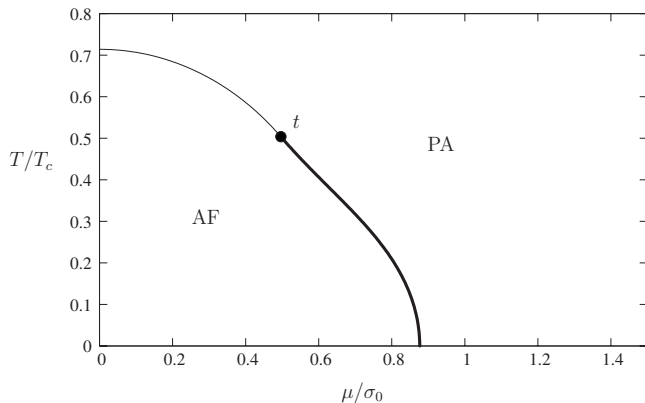


FIG. 6. Phase diagram for  $t/t_{\text{TH}}=0.7$  and  $\alpha=0.05$ . The transition from the AF to the PA is second order at half filling (thin line) and first order at zero temperature. These two lines merge at a tricritical point,  $t$ . The scales  $T_c$  and  $\sigma_0$  are those defined in Fig. 2.

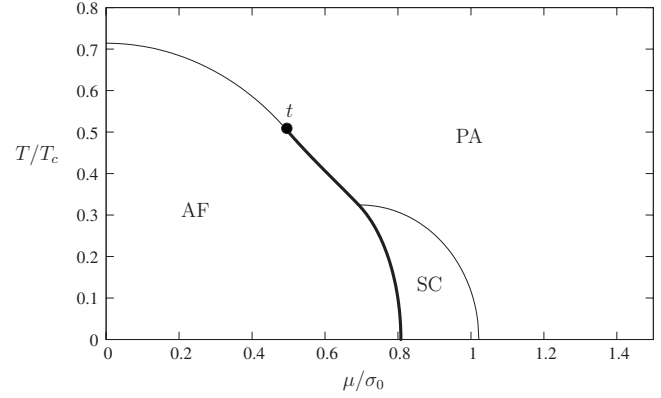


FIG. 7. Phase diagram for  $t/t_{\text{TH}}=0.7$  and  $\alpha=0.2$ . Thin lines are second order and the thick line is first order;  $t$  is a tricritical point. The scales  $T_c$  and  $\sigma_0$  are those defined in Fig. 2.

order to understand its influence on the phase structure of the system. Figures 6–8 show the phase diagrams that are realized by a system with  $t/t_{\text{TH}}=0.7$  and  $\alpha=0.05, 0.2$ , and  $0.8$ , respectively. The resulting phase diagrams exhibit the same basic topologies found in the previous case. No additional phase diagrams are introduced by a larger value of  $t$ . Increasing the hopping term has two primary effects. First, the threshold parameters  $\alpha_\Delta$  and  $\alpha_b$  are both reduced. Second, larger  $t$  tends to increase the region occupied by the superconducting phase and decrease that occupied by the antiferromagnetic phase. This can be understood directly from the form of the quasiparticle energies  $\varepsilon_\pm$  in Eq. (36). In the antiferromagnetic phase,  $\varepsilon_\pm$  directly depends on the combination  $\sigma^2 + t^2$ . Increasing  $t$  without excessively increasing  $\varepsilon_\pm$  thus requires a reduction in  $\sigma$ . In the superconducting phase, however,  $\varepsilon_\pm$  varies with  $t$  as  $(t \pm \mu)^2 + |\Delta|^2$  so that the  $\mu$  term tends to reduce the sensitivity of  $\Delta$  with respect to variations of  $t$ .

### 3. Absence of a mixed broken-symmetry phase

In Sec. IV C, we found that  $\Omega$  has a series expansion of the form  $\Omega(\sigma, \Delta) \approx \Omega(0, 0) + a_4 |\Delta|^4 + b_4 \sigma^4 + c_4 \sigma^2 |\Delta|^2$  near the bicritical point, where the coefficients  $a_4$ ,  $b_4$ , and  $c_4$  are

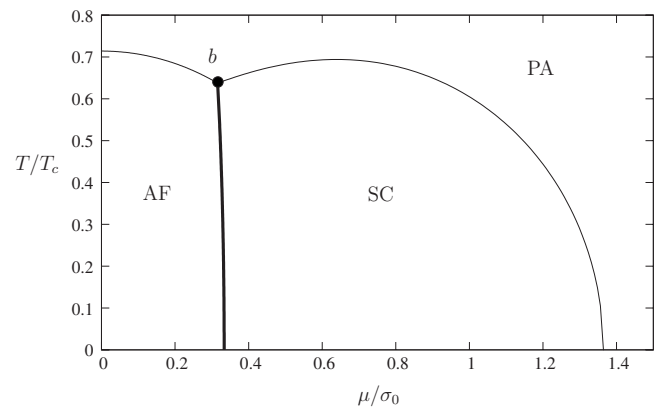


FIG. 8. Phase diagram for  $t/t_{\text{TH}}=0.7$  and  $\alpha=0.8$ . Thin lines are second order and the thick line is first order.  $b$  is a bicritical point. The scales  $T_c$  and  $\sigma_0$  are those defined in Fig. 2.

known functions of  $t$  and  $\alpha$  which satisfy the inequality  $4a_4b_4 - c_4^2 < 0$ . As a result, a phase with both nonzero  $\sigma$  and nonzero  $\Delta$  is never realized. Such a mixed broken-symmetry state is in fact a solution of the coupled gap equations  $\partial\Omega/\partial\sigma=0$  and  $\partial\Omega/\partial\Delta=0$  but is unstable since the matrix of second derivatives has a negative determinant [i.e., the mixed broken-symmetry phase is necessarily a saddle point of  $\Omega(\sigma, \Delta)$ ]. In this regard, the present random matrix result differs from mean-field results for both the Hubbard<sup>29,33</sup> and  $t$ - $J$  models,<sup>30</sup> for which mixed broken-symmetry phases are found along with a tetracritical—rather than a bicritical—point in the  $(\mu, T)$  plane. Coexisting phases have also been reported in a number of numerical studies including variational cluster perturbation theory<sup>34</sup> and variational Monte Carlo methods,<sup>35</sup> while other studies report phase separation.<sup>36</sup> Calculations from cellular dynamical mean-field theory show a strong tendency to a homogeneous coexistence phase at weak coupling and a first-order phase transition at strong coupling.<sup>37</sup>

The differences arising at mean-field level between the random matrix approach and microscopic models may well be a consequence of the neglect of density of states effects in the potential of Eq. (31). In the absence of a logarithmic divergence in the gap equations, condensation fields are weakened and the energy balance between the phases can be upset. We note, however, that some authors have questioned whether the phase diagram is necessarily controlled solely by the Van Hove singularity in the bare density of states<sup>35</sup> as mean-field microscopic results would seem to show. The discussion of this point shows an example of results that should not be considered universal or robust; further careful numerical studies are needed to settle this issue. Although a random matrix model will not give a definite answer, its comparison to other mean-field approaches can help in identifying those features that are sensitive to the specifics of models and their numerical treatment and which are thus not protected by the symmetries.

#### 4. Absence of an exact higher symmetry

It is interesting to ask whether the potential of Eq. (31) can exhibit a higher symmetry, such as the SO(5) symmetry proposed by Zhang,<sup>38</sup> either exactly or approximately in certain regions of the phase diagram. Such a symmetry should manifest itself in the possibility of writing  $\Omega$  as a function of a single combination of the condensation fields, such as  $\sigma^2 + |\Delta|^2$ .

If we restrict our attention to  $\alpha=B/A < 1$  so that AF order truly competes with SC order, the answer is negative. Near the bicritical point, we found that the coefficients in the series expansion of the thermodynamical potential,  $\Omega \approx \Omega(0,0) + a_4|\Delta|^4 + b_4\sigma^4 + c_4|\Delta|^2\sigma^2$ , satisfy the inequality  $4a_4b_4 - c_4^2 < 0$ . This inequality eliminates the possibility that the fourth-order terms are a perfect square. Therefore, no simple symmetry—allowing for a rotation of the AF order parameter into the SC order parameter—can be identified at the level of two-body correlators. Moreover, we have been unable to identify any other symmetry from an expansion of  $\Omega$  to higher order.

The limit  $\alpha \rightarrow 1$  can roughly be seen as one of approximate higher symmetry. In fact, the bicritical point migrates to the vertical axis,  $\mu_b \rightarrow 0$ , and we have  $4a_4b_4 - c_4^2 \propto \mu_b^4 \rightarrow 0$ . Then,  $\Omega$  can be approximated as  $\Omega \approx \Omega(0,0) + (\sqrt{a_4}|\Delta|^2 + \sqrt{b_4}\sigma^2)^2$ , with  $a_4 \approx b_4$ , a form which shows an approximate higher symmetry. In the limit  $\alpha \rightarrow 1$ , the antiferromagnetic phase exists only in the immediate vicinity of half filling and the superconducting phase dominates the region of finite chemical potential. Realizing such a situation in actual materials would require very specific relationships between the coupling constants.

## V. CONCLUSIONS

In this paper, we have suggested a mean-field model for investigating the thermodynamic competition between magnetic and superconducting orders in a two-dimensional square lattice. This model describes interactions at a more microscopic level than the familiar Hubbard or  $t$ - $J$  Hamiltonian through the introduction of density and spin-fluctuation exchanges. The single-particle Hamiltonian is given a block structure that is dictated by spin, time-reversal, and bipartite symmetries, and the detailed dynamics of interactions are replaced by a normal distribution of random matrix elements. The model is formulated in a momentum representation so that a coarse description of the first Brillouin region allows us to introduce a  $d$ -wave form factor that possesses the appropriate sign symmetry but neglects detailed momentum dependences. Although this approach may seem elaborate at first sight, the model simplifies dramatically as one proceeds through the derivation. In fact, the resulting thermodynamic potential has a simple and well-defined structure that depends solely on the form of the quasiparticle energies in given condensation fields.

We have explored a number of physically relevant cases for which the interactions are attractive in both the antiferromagnetic and the  $d$ -wave channels and repulsive in the  $s$ -wave channel. Such interactions naturally place greater weight on spin-fluctuation exchanges, particularly those involving a momentum transfer  $\sim \mathbf{Q}$ . The phase diagram is found to depend on a single parameter ratio,  $\alpha$ , and a limited number of topologies appear as a function of  $\alpha$ . None of these topologies allows for a mixed broken-symmetry phase with coexisting antiferromagnetism and superconductivity, probably as a consequence of the absence of the singularity in the bare density of states. Except for the smallest values of  $\alpha$ , which result in a phase diagram involving antiferromagnetism alone, the range of values for  $\alpha$  that correspond to a given topology is rather large. As a result, these phase topologies should be regarded as robust with respect to moderate variations of the detailed description of the interactions.

The random matrix approach described here has a broad range of applications, as we have demonstrated by studying the phase diagrams of QCD and that of the cuprates. Given the relatively simple structure that is obtained for the thermodynamic potential, this method is convenient for obtaining a direct zeroth-order description of the phase structure. In fact, such a potential could also be written immediately given only knowledge of the quasiparticle energies. How-

ever, the explicit construction of the microscopic interactions carries additional and possibly useful information regarding the dependence of the coupling parameters on microscopic variances. In the vicinity of critical points, the thermodynamic potential can be expanded as a power series of the condensation fields in a form similar to a Ginsburg-Landau theory. The information relating microscopic processes to the coupling parameters thus imposes significant constraints on the Ginzburg-Landau coefficients and offers an improved understanding of the relation between global properties of the system and its microscopic description.

In some sense, this paper can be regarded as “open source theoretical physics.” It contains an “algorithm” that can be adapted for use in other problems. The shortcoming if this approach is clearly that it is no more than mean-field theory. Its merit lies in the fact that it gives results which are averaged over an ensemble of theories. Model-dependent details are thus eliminated. The resulting thermodynamic potential is dictated by the underlying symmetries of the problem and is likely to be robust.

Here, we have concentrated on the possible topologies of the phase diagram and hence on the global minimum of  $\Omega$  as a function of  $\mu$  and  $T$ . We believe that the results presented here represent a generic mean-field phase diagram. Other features found either in theory or experiment are likely to be “fragile.” For instance, the absence of a mixed broken-symmetry state in the present model suggests that this state is sensitive to model-dependent details, numerical approximations, or to the detailed properties of the sample studied. Indeed, the mixed broken-symmetry state is not generic in the cuprates.<sup>37</sup> More comprehensive studies of  $\Omega$  can be useful in revealing, e.g., the competition between local minima in  $\Omega$  and other relatively fragile structures that could be sensitive to model-dependent details and numerical approximations. In this sense, we suggest that the present methods may provide a useful complement to the investigation of detailed models of these systems.

#### APPENDIX: CALCULATION OF THE THERMODYNAMIC POTENTIAL

In this appendix, we sketch the calculations leading to the thermodynamic potential,  $\Omega$ , of Eq. (31). The calculations closely follow those of the random matrix model for QCD given in Ref. 23.

We start with the single-fermion Hamiltonian of Eq. (23) and integrate  $\exp(-\psi^\dagger H_{\text{int}} \psi)$  over the matrix elements of  $H_{\text{int}}$  according to the normal probability distribution  $P(H_{\text{int}})$  of Eq. (25). This integration can be written as

$$\int DH_{\text{int}} P(H_{\text{int}}) e^{-\psi^\dagger H_{\text{int}} \psi} = e^Y, \quad (\text{A1})$$

where the four-fermion potential,  $Y$ , receives a contribution from each of the eight independent block matrices

$$Y = Y_{B0d} + Y_{\mathbf{B}_d} + Y_{B0o} + Y_{\mathbf{B}_o} + Y_{C0d} + Y_{\mathbf{C}_d} + Y_{C0o} + Y_{\mathbf{C}_o}. \quad (\text{A2})$$

An integration over the matrix elements describing density fluctuations gives

$$Y_{B0d} = \frac{1}{32M\Sigma_{B0d}^2} \sum_{ij} (1_i 1_j + 2_i 2_j + 3_j 3_i + 4_j 4_i) (\text{H.c.}), \quad (\text{A3})$$

$$Y_{B0o} = \frac{1}{32M\Sigma_{B0o}^2} \sum_{ij} (1_i 2_j + 2_i 1_j + 3_j 4_i + 4_j 3_i) (\text{H.c.}), \quad (\text{A4})$$

$$Y_{C0d} = \frac{1}{32M\Sigma_{C0d}^2} \sum_{ij} (1_i 3_j + 2_i 4_j + 1_j 3_i + 2_j 4_i) (\text{H.c.}), \quad (\text{A5})$$

$$Y_{C0o} = \frac{1}{32M\Sigma_{C0o}^2} \sum_{ij} (1_i 4_j + 2_j 3_i + 1_i 4_j + 2_i 3_j) (\text{H.c.}), \quad (\text{A6})$$

where (H.c.) indicates Hermitian conjugation and  $1_i 1_j$  is a compact notation for  $\sum_\alpha \psi_{1i\alpha}^\dagger \psi_{1j\alpha}$ . (Here,  $\psi_{1j\alpha}$  represents a state with the momentum label  $j$  in region 1 and a spin index  $\alpha$ .) Integration over the matrix elements of the blocks describing spin fluctuations produces the terms

$$Y_{\mathbf{B}_d} = \frac{1}{32M\Sigma_{\mathbf{B}_d}^2} \sum_{ij} (1_i \boldsymbol{\sigma} 1_j + 2_i \boldsymbol{\sigma} 2_j + 3_j \boldsymbol{\sigma} 3_i + 4_j \boldsymbol{\sigma} 4_i) \cdot (\text{H.c.}), \quad (\text{A7})$$

$$Y_{\mathbf{B}_o} = \frac{1}{32M\Sigma_{\mathbf{B}_o}^2} \sum_{ij} (1_i \boldsymbol{\sigma} 2_j + 2_i \boldsymbol{\sigma} 1_j + 3_j \boldsymbol{\sigma} 4_i + 4_j \boldsymbol{\sigma} 3_i) \cdot (\text{H.c.}), \quad (\text{A8})$$

$$Y_{\mathbf{C}_d} = \frac{1}{32M\Sigma_{\mathbf{C}_d}^2} \sum_{ij} (1_i \boldsymbol{\sigma} 3_j + 2_i \boldsymbol{\sigma} 4_j + 1_j \boldsymbol{\sigma} 3_i + 2_j \boldsymbol{\sigma} 4_i) \cdot (\text{H.c.}), \quad (\text{A9})$$

$$Y_{\mathbf{C}_o} = \frac{1}{32M\Sigma_{\mathbf{C}_o}^2} \sum_{ij} (1_i \boldsymbol{\sigma} 4_j + 2_i \boldsymbol{\sigma} 3_j + 1_j \boldsymbol{\sigma} 4_i + 2_j \boldsymbol{\sigma} 3_i) \cdot (\text{H.c.}), \quad (\text{A10})$$

where the notation  $1_i \boldsymbol{\sigma} 2_j$  stands for  $\sum_{\alpha\beta} \psi_{1i\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} \psi_{2j\beta}$ .

Next, fermion fields are rearranged to make the condensation channels apparent. Schematically, terms of the form  $\psi_i^\dagger \psi_j \psi_j^\dagger \psi_i$  (with the sum over  $i$  and  $j$  implied) are brought into the form  $-\psi_i^\dagger \psi_i \psi_j^\dagger \psi_j$ , which gives rise to condensates  $\sim \psi_i^\dagger \psi_i$  that are relevant for antiferromagnetism. Similarly, terms of the form  $\psi_i^\dagger \psi_j \psi_i^\dagger \psi_j$  are rewritten as  $\psi_i^\dagger \psi_i \psi_j^\dagger \psi_j$ , which contain condensates of the form  $\sim \psi_j \psi_j$  and are relevant for superconductivity. Uncrossing necessitates keeping track of momentum and spin indices. To uncross spin indices, we use the SU(2) Fierz identities

$$\delta_{ab} \delta_{cd} = \frac{1}{2} \delta_{ad} \delta_{cb} + \frac{1}{2} \boldsymbol{\sigma}_{ad} \cdot \boldsymbol{\sigma}_{cb}, \quad (\text{A11})$$



$$\boldsymbol{\sigma}_{ab} \cdot \boldsymbol{\sigma}_{cd} = \frac{3}{2} \delta_{ad} \delta_{cb} - \frac{1}{2} \boldsymbol{\sigma}_{ad} \cdot \boldsymbol{\sigma}_{cb} \quad (\text{A12})$$

for the antiferromagnetism channel and the identities

$$\delta_{ab} \delta_{cd} = \frac{1}{2} \delta_{ac} \delta_{db} + \frac{1}{2} \boldsymbol{\sigma}_{ac} \cdot \boldsymbol{\sigma}_{db}, \quad (\text{A13})$$

$$\begin{aligned} \boldsymbol{\sigma}_{ab} \cdot \boldsymbol{\sigma}_{cd} = & \frac{1}{2} \delta_{ac} \delta_{db} + \frac{1}{2} (\boldsymbol{\sigma}_1)_{ac} (\boldsymbol{\sigma}_1)_{db} - \frac{3}{2} (\boldsymbol{\sigma}_2)_{ac} (\boldsymbol{\sigma}_2)_{db} \\ & + \frac{1}{2} (\boldsymbol{\sigma}_3)_{ac} (\boldsymbol{\sigma}_3)_{db} \end{aligned} \quad (\text{A14})$$

for superconductivity channels. To uncross momentum indices, each quadratic term  $\psi_r^\dagger \psi_s$  (where  $r$  and  $s$  now denote momentum indices) is written as an element of a  $4 \times 4$  momentum matrix which is then decomposed on a complete basis of 16 Hermitian Dirac matrices<sup>23</sup> that satisfy

$$\text{Tr}(\Gamma_k \Gamma_l) = 4 \delta_{kl} \quad (k, l = 1, \dots, 16). \quad (\text{A15})$$

There exists a representation in which this basis contains the operators  $\Gamma_{\text{AF}}$ ,  $\Gamma_{\text{SC-}d}$ , and  $\Gamma_{\text{SC-}s}$ , which were introduced, respectively, in Eqs. (8), (10), and (11), as well as an operator that is proportional to the hopping term,  $\Gamma_r$ , of Eq. (28). With this complete basis defined, we uncross momentum according to the Fierz identities

$$(\Gamma_k)_{ab} (\Gamma_l)_{cd} = \sum_{mn} x_{klmn} (\Gamma_m)_{ad} (\Gamma_n)_{cb}, \quad (\text{A16})$$

$$x_{klmn} = \frac{1}{16} \text{Tr}(\Gamma_k \Gamma_n \Gamma_l \Gamma_m) \quad (\text{A17})$$

for terms relevant for antiferromagnetism and

$$(\Gamma_k)_{ab} (\Gamma_l)_{cd} = \sum_{mn} x_{klmn} (\Gamma_m)_{ac} (\Gamma_n)_{db}, \quad (\text{A18})$$

$$x_{klmn} = \frac{1}{16} \text{Tr}(\Gamma_k \Gamma_n \Gamma_l^T \Gamma_m) \quad (\text{A19})$$

for terms relevant for superconductivity. Here,  $\text{Tr}$  denotes a trace and  $\Gamma^T$  is the transpose of  $\Gamma$ .

Restricting ourselves to the antiferromagnetic and superconducting channels, uncrossing gives us the four-fermion interaction

$$Y = Y_{\text{AF}} + Y_{\text{SC-}d} + \dots \quad (\text{A20})$$

$$\begin{aligned} = & a_{\text{AF}} (\psi^\dagger \Gamma_{\text{AF}} \sigma_3 \psi)^2 + a_{\text{SC-}s} (\psi^\dagger \Gamma_{\text{SC-}s} \sigma_2 \psi^\dagger) (\psi \Gamma_{\text{SC-}s} \sigma_2 \psi) \\ & + a_{\text{SC-}d} (\psi^\dagger \Gamma_{\text{SC-}d} \sigma_2 \psi^\dagger) (\psi \Gamma_{\text{SC-}d} \sigma_2 \psi), \end{aligned} \quad (\text{A21})$$

where a summation over spin, momentum, and random matrix indices is implied in each fermion bilinear. The coefficients in the respective channels are given by

$$\begin{aligned} a_{\text{AF}} = & \frac{1}{256M} \left( -\frac{1}{\Sigma_{B0d}^2} - \frac{1}{\Sigma_{B0o}^2} - \frac{1}{\Sigma_{C0d}^2} - \frac{1}{\Sigma_{C0o}^2} + \frac{1}{\Sigma_{\mathbf{B}_d}^2} + \frac{1}{\Sigma_{\mathbf{B}_o}^2} \right. \\ & \left. + \frac{1}{\Sigma_{\mathbf{C}_d}^2} + \frac{1}{\Sigma_{\mathbf{C}_o}^2} \right), \end{aligned} \quad (\text{A22})$$

$$\begin{aligned} a_{\text{SC-}s} = & \frac{1}{256M} \left( \frac{1}{\Sigma_{B0d}^2} + \frac{1}{\Sigma_{B0o}^2} + \frac{1}{\Sigma_{C0d}^2} + \frac{1}{\Sigma_{C0o}^2} - \frac{3}{\Sigma_{\mathbf{B}_d}^2} - \frac{3}{\Sigma_{\mathbf{B}_o}^2} \right. \\ & \left. - \frac{3}{\Sigma_{\mathbf{C}_d}^2} - \frac{3}{\Sigma_{\mathbf{C}_o}^2} \right), \end{aligned} \quad (\text{A23})$$

$$\begin{aligned} a_{\text{SC-}d} = & \frac{1}{256M} \left( \frac{1}{\Sigma_{B0d}^2} - \frac{1}{\Sigma_{B0o}^2} + \frac{1}{\Sigma_{C0d}^2} - \frac{1}{\Sigma_{C0o}^2} - \frac{3}{\Sigma_{\mathbf{B}_d}^2} + \frac{3}{\Sigma_{\mathbf{B}_o}^2} \right. \\ & \left. - \frac{3}{\Sigma_{\mathbf{C}_d}^2} + \frac{3}{\Sigma_{\mathbf{C}_o}^2} \right). \end{aligned} \quad (\text{A24})$$

Here, a positive (negative) coefficient corresponds to an attractive (repulsive) channel. From the above equations, we thus see that interactions for which the inverse variances  $\Sigma_{\mathbf{B}_d}^2$ ,  $\Sigma_{\mathbf{B}_o}^2$ ,  $\Sigma_{\mathbf{C}_d}^2$ , or  $\Sigma_{\mathbf{C}_o}^2$  are small compared to  $\Sigma_{B0d}^2$ ,  $\Sigma_{B0o}^2$ ,  $\Sigma_{C0d}^2$ , or  $\Sigma_{C0o}^2$  are attractive in the antiferromagnetic channel. Such interactions favor the exchange of spin fluctuations over the exchange of density fluctuations. Similarly, interactions can be made attractive in the  $d$ -wave channel and repulsive in the  $s$ -wave channel by favoring the elements of the off-diagonal block matrices  $\mathbf{B}_o$  and  $\mathbf{C}_o$  over those of the diagonal blocks  $\mathbf{B}_d$  and  $\mathbf{C}_d$ . Such interactions favor the exchange of spin fluctuations with a large momentum transfer,  $\sim \mathbf{Q}$ .

Below, we will assume that the random interactions favor antiferromagnetism and superconductivity in the  $d$ -wave channel and are repulsive for  $s$ -wave pairs; this situation corresponds to a particular choice for the variances such that  $a_{\text{AF}} > 0$  and  $a_{\text{SC-}d} > 0$ , whereas  $a_{\text{SC-}s} < 0$ . We will thus neglect the  $s$ -wave channel in the remainder of the calculations. In this case, combining Eqs. (12) and (A1) yields a partition function of the form

$$Z(\mu, T) = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi e^{-\psi^\dagger H_0 \psi + Y}, \quad (\text{A25})$$

where, according to Eqs. (A20) and (A21),

$$Y \sim a_{\text{AF}} (\psi^\dagger \Gamma_{\text{AF}} \sigma_3 \psi)^2 + a_{\text{SC-}d} (\psi^\dagger \Gamma_{\text{SC-}d} \sigma_2 \psi^\dagger) (\psi \Gamma_{\text{SC-}d} \sigma_2 \psi). \quad (\text{A26})$$

The quartic fermion terms can now be written as the difference of two squares. Each square is linearized by the use of a Hubbard-Stratonovitch transformation

$$e^{A Q^2} \sim \int dx \exp\left(-\frac{x^2}{4A} - Qx\right), \quad (\text{A27})$$

which introduces an auxiliary field  $x$ . When applied to Eq. (A26), such transformations introduce a real field,  $\sigma$ , to be associated with antiferromagnetism and a complex field,  $\Delta$ , to be related to superconductivity. The partition function is then written as

$$Z(\mu, T) \sim \int d\sigma d\Delta d\Delta^* \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp\left(-\psi^\dagger H_0 \psi - \frac{|\Delta|^2}{4a_{\text{SC}-d}} - \frac{\sigma^2}{4a_{\text{AF}}}\right) \exp\left(-\sigma\psi^\dagger \Gamma_{\text{AF}} \sigma_3 \psi - \frac{\Delta}{2} \psi^\dagger \Gamma_{\text{SC}-d} \sigma_2 \psi^\dagger - \frac{\Delta^*}{2} \psi \Gamma_{\text{SC}-d} \sigma_2 \psi\right), \quad (\text{A28})$$

where  $H_0 = -\mu + \Omega_T + \Gamma_t$  [see Eq. (26)]. Using the spinor  $\Psi = (\psi_\uparrow, \psi_\downarrow)$ , the fermion bilinears can be arranged in the Gor'kov form  $\Psi^\dagger \tilde{H} \Psi$ , with

$$\tilde{H} = \begin{pmatrix} -\mu + \Omega_T + \Gamma_t + \sigma \Gamma_{\text{AF}} & -i\Delta \Gamma_{\text{SC}-d} \\ i\Delta^* \Gamma_{\text{SC}-d} & \mu - \Omega_T - \Gamma_t + \sigma \Gamma_{\text{AF}} \end{pmatrix}. \quad (\text{A29})$$

Then, integrating over the fermion fields gives

$$\int \mathcal{D}\psi^\dagger \mathcal{D}\psi e^{-\Psi^\dagger \tilde{H} \Psi} = \text{Det}[\tilde{H}], \quad (\text{A30})$$

so that the partition function in Eq. (A28) can be written as

$$Z(\mu, T) = \int d\sigma d\Delta d\Delta^* e^{-8M\Omega(\sigma, \Delta)}, \quad (\text{A31})$$

where  $\Omega$  is the thermodynamic potential

$$\Omega = \frac{1}{32M} \left( \frac{|\Delta|^2}{a_{\text{SC}-d}} + \frac{\sigma^2}{a_{\text{AF}}} - 4 \log \text{Det}(\tilde{H}) \right). \quad (\text{A32})$$

Calculating the determinant of  $\tilde{H}$  in Eq. (A29) then gives

$$\Omega(\sigma, \Delta) = A|\Delta|^2 + B\sigma^2 - \frac{1}{4} \log[(\sqrt{\sigma^2 + t^2} - \mu)^2 + |\Delta|^2 + T^2] - \frac{1}{4} \log[(\sqrt{\sigma^2 + t^2} + \mu)^2 + |\Delta|^2 + T^2], \quad (\text{A33})$$

with

$$A = 8 \left( \frac{1}{\Sigma_{B0d}^2} - \frac{1}{\Sigma_{B0o}^2} + \frac{1}{\Sigma_{C0d}^2} - \frac{1}{\Sigma_{C0o}^2} - \frac{3}{\Sigma_{\mathbf{B}_d}^2} + \frac{3}{\Sigma_{\mathbf{B}_o}^2} - \frac{3}{\Sigma_{\mathbf{C}_d}^2} + \frac{3}{\Sigma_{\mathbf{C}_o}^2} \right)^{-1}, \quad (\text{A34})$$

$$B = 8 \left( -\frac{1}{\Sigma_{B0d}^2} - \frac{1}{\Sigma_{B0o}^2} - \frac{1}{\Sigma_{C0d}^2} - \frac{1}{\Sigma_{C0o}^2} + \frac{1}{\Sigma_{\mathbf{B}_d}^2} + \frac{1}{\Sigma_{\mathbf{B}_o}^2} + \frac{1}{\Sigma_{\mathbf{C}_d}^2} + \frac{1}{\Sigma_{\mathbf{C}_o}^2} \right)^{-1}. \quad (\text{A35})$$

We can evaluate the properties of the competing phases by determining the minima of the potential  $\Omega$ . In the thermodynamic limit, where  $M$  is taken to infinity, these minima will

give the exact solutions for the system since a saddle-point evaluation gives  $\lim_{M \rightarrow \infty} (1/8M) \ln Z = -\min_{\sigma, \Delta}(\Omega)$ .

We mentioned earlier an alternative to the bipartite symmetry. This alternative choice consists in taking the matrix elements between states  $\mathbf{p}_1 + \mathbf{Q}$  and  $\mathbf{p}_2 + \mathbf{Q}$  equal to those between  $\mathbf{p}_1$  and  $\mathbf{p}_2$ , so that Eq. (22) becomes

$$H_\mu = \begin{pmatrix} D_\mu & E_\mu \\ E_\mu^\dagger & D_\mu \end{pmatrix}, \quad (\text{A36})$$

where now  $E_\mu$  are complex. Such a choice modifies the four-fermion potentials  $Y_{B0o}$ ,  $Y_{C0o}$ ,  $Y_{\mathbf{B}_o}$ , and  $Y_{\mathbf{C}_o}$  as

$$Y_{B0o} = \frac{1}{8M\Sigma_{B0o}^2} \sum_{ij} (1_i 2_j + 4_j 3_i) (\text{H.c.}), \quad (\text{A37})$$

$$Y_{C0o} = \frac{1}{8M\Sigma_{C0o}^2} \sum_{ij} (1_i 4_j + 2_j 3_i) (\text{H.c.}), \quad (\text{A38})$$

$$Y_{\mathbf{B}_o} = \frac{1}{8M\Sigma_{\mathbf{B}_o}^2} \sum_{ij} (1_i \sigma_{2j} + 4_j \sigma_{3i}) \cdot (\text{H.c.}), \quad (\text{A39})$$

$$Y_{\mathbf{C}_o} = \frac{1}{8M\Sigma_{\mathbf{C}_o}^2} \sum_{ij} (1_i \sigma_{4j} + 2_j \sigma_{3i}) \cdot (\text{H.c.}). \quad (\text{A40})$$

The coupling constants  $a_{\text{AF}}$ ,  $a_{\text{SC}-s}$ , and  $a_{\text{SC}-d}$  become

$$a_{\text{AF}} = \frac{1}{256M} \left( -\frac{1}{\Sigma_{B0d}^2} - \frac{1}{\Sigma_{C0d}^2} + \frac{1}{\Sigma_{\mathbf{B}_d}^2} + \frac{1}{\Sigma_{\mathbf{C}_d}^2} \right), \quad (\text{A41})$$

$$a_{\text{SC}-s} = \frac{1}{256M} \left( \frac{1}{\Sigma_{B0d}^2} + \frac{2}{\Sigma_{B0o}^2} + \frac{1}{\Sigma_{C0d}^2} + \frac{2}{\Sigma_{C0o}^2} - \frac{3}{\Sigma_{\mathbf{B}_d}^2} - \frac{6}{\Sigma_{\mathbf{B}_o}^2} - \frac{3}{\Sigma_{\mathbf{C}_d}^2} - \frac{6}{\Sigma_{\mathbf{C}_o}^2} \right), \quad (\text{A42})$$

$$a_{\text{SC}-d} = \frac{1}{256M} \left( \frac{1}{\Sigma_{B0d}^2} - \frac{2}{\Sigma_{B0o}^2} + \frac{1}{\Sigma_{C0d}^2} - \frac{2}{\Sigma_{C0o}^2} - \frac{3}{\Sigma_{\mathbf{B}_d}^2} + \frac{6}{\Sigma_{\mathbf{B}_o}^2} - \frac{3}{\Sigma_{\mathbf{C}_d}^2} + \frac{6}{\Sigma_{\mathbf{C}_o}^2} \right). \quad (\text{A43})$$

Again, the interaction can be made attractive in the AF channel by favoring spin over density exchanges, while it is attractive in the  $d$ -wave channel and repulsive for  $s$ -wave pairing when large momentum transfers are favored. The resulting thermodynamic potential has the form of Eq. (A33), although with slightly different expressions for  $A$  and  $B$ . The main results in the text remain valid with this alternative choice.

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- <sup>40</sup>Expressed here in the momentum basis, the matrix  $H_\mu$  of size  $N \times N$  contains  $N^2/4$  independent elements for  $B_\mu$  and  $N^2/4 + N/2$  independent elements for  $C_\mu$ , hence a total of  $N(N+1)/2$  independent elements. This is also the number of independent elements found in the more familiar coordinate representation, where  $H_\mu$  is real symmetric.
- <sup>41</sup>As is usually the case in random matrix theory, the technically convenient choice of a Gaussian distribution is completely passive. All final results will depend only on the inverse variances,  $\Sigma_b^2$ , and identical results would be obtained for other choices of  $P(H_{\text{int}})$  with equal variances.